A Regular Isotopy Alexander Theorem for Thompson's Group F

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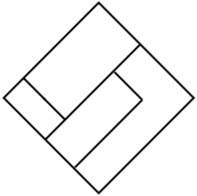
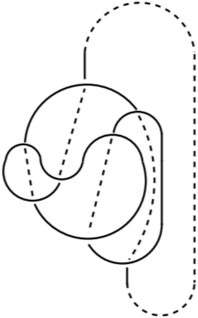
Constructing Links from Elements of the Thompson Group

Recall that each element of the Thompson group can be represented by a pair of binary trees, one right side up and one upside down, stacked on top of each other with their leaves attached in order.

If $g \in F$, then let $L(g)$ denote the associated link, which is constructed as follows:

$g =$ 

$L(g) =$ 

Not all links are regular isotopic to some $L(g)$

The parity of the number of crossings on a link diagram is invariant under regular isotopy.

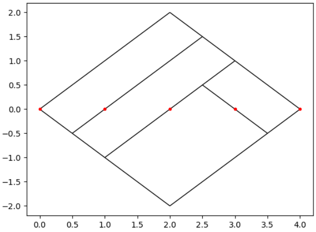
For $g \in F$, $L(g)$ has an even number of crossings.

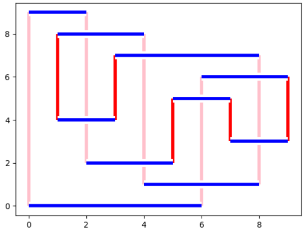
This trefoil is not regular isotopic to any $L(g)$:

A Stronger Restriction

Call a link diagram D *compliant* if for every component C of D, the number of crossings for which C contains the understrand must be even.

For any $g \in F$, $L(g)$ is compliant.

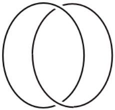




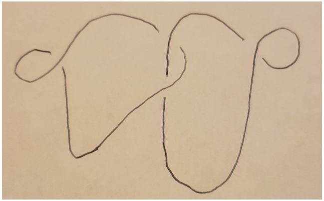
Compliance is invariant under regular isotopy!

An example and a non-example

Despite having an even number of crossings, this Hopf link is not compliant.



If we add a couple of twists, we get a compliant Hopf link.



In fact, this Hopf link is regular isotopic to the one on the previous slide!

The characterization: Part 1

Theorem

Let D be a link diagram. D is compliant if and only if there is a $g \in F$ such that D and $L(g)$ are regular isotopic.

We have already proved one direction of the theorem: compliance is invariant under regular isotopy, and $L(g)$ is always compliant, so if D is regular isotopic to $L(g)$, it must be compliant.

The characterization: Part B

We will sketch the proof of the other direction.

Let D be a compliant link diagram. By Jones's Alexander theorem for F, we can find $h \in F$ such that $L(h)$ is equivalent to D.

We wish to find $g \in F$ such that $L(g)$ is equivalent to $L(h)$ (and therefore to D), and such that for each component C of $L(h)$, the writhe and winding number of C are the same as the writhe and winding number of the corresponding component of D.

Then, we are done!

Adjusting the writhes and winding numbers of components of $L(g)$

If you know the equivalence class of a compliant link, you can figure out the parity of the number of self-crossings of each component.

If you know the parity of the number of self-crossings of each component, you can deduce the parities of the writhe and winding number of each component.

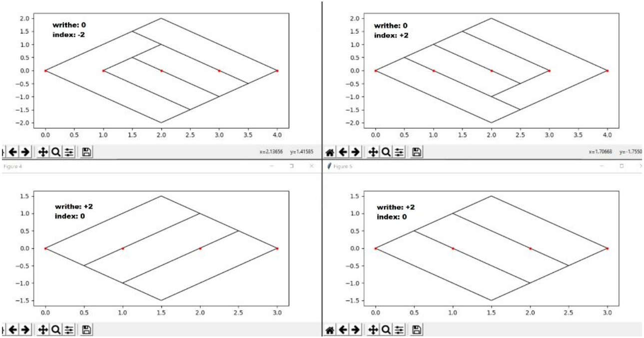
Thus, if we can add our choice from $(2, 0)$, $(0, 2)$, $(-2, 0)$, $(0, -2)$ to the writhe or winding number of a component, while fixing the writhes and winding numbers of the other components, we are done.

The Final Reduction, and a Shoutout to Computers

In fact, this problem reduces further to finding elements of F that give the unknot, with (writhe, winding number -1) equal to each of $(2,0),(0,2),(-2,0),(0,-2)$.

This is related to *connected sums*.

Here are those elements, found using a computer search!



References

Alexander Coward. Ordering the reidemeister moves of a classical knot. *Algebraic Geometric Topology*, 6(2):659–671, May 2006. ISSN 1472-2747. doi: 10.2140/agt.2006.6.659. URL http://dx.doi.org/10.2140/agt.2006.6.659.

Vaughan F. R. Jones. On the construction of knots and links from thompson's groups. *arXiv: Geometric Topology*, 2016.